Bessel Functions and Cylindrical Geometry

Steady state temperature distribution in a semi-infinite cylinder. The energy balance in cylindrical coordinates:

\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \]

**Boundary Conditions:**

- \( T(1, z) = 0 \)
- \( T(0, z) \) finite
- \( T(r, 0) = f(r) \)
- \( T(r, z) \to 0 \) as \( z \to \infty \)

Assume a separation of variables solution exists:
(can be shown using boundary conditions & Sturm-Liouville Thm)

\[ T(r,z) = R(r) Z(z) \]

hence

\[ Z \frac{d^2 R}{dr^2} + \frac{Z}{r} \frac{dR}{dr} + R \frac{d^2 T}{dz^2} = 0 \]

Divide by \( RZ \) to get (primes denote differentials)

\[ \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\frac{Z''}{Z} = -\lambda^2 \]

\[ \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\lambda^2 \] (chosen to give exponentials in \( Z \) directions)

\[ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \lambda^2 R = 0 \]

or

\[ r \frac{d^2 R}{dr^2} + \frac{dR}{dr} + r \lambda^2 R = 0 \]

\[ \frac{d}{dr} \left( r \frac{dR}{dr} \right) + r \lambda^2 R = 0 \]

Remember \( S-L \) Equation

\[ \frac{d}{dx} \left[ p(x) \frac{dy}{dx} \right] - s(x) y + \lambda^2 r(x) y = 0 \]
Clearly our equation is a SL equation:

\[ p(x) = r \]
\[ s(x) = 0 \]
\[ r(x) = r \quad \Leftarrow \text{weighting function} \]

Remember that if the B.C.’s are appropriate, the solutions of this equation will be orthogonal eigenfunctions w.r.t the weight function \( r \)

\[ \downarrow \text{wt. function} \]

\[ \int_0^1 R_n \left( \lambda_n r \right) R_m \left( \lambda_m r \right) r \, dr = \delta_{mn} \int_0^1 R_m^2 \left( \lambda_m r \right) r \, dr \]

We can also show that \( R(r) \) is the well-known Bessel function:

\[ r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + r^2 \lambda^2 R = 0 \]

Let \( x = \lambda r \Rightarrow x / \lambda = r \)
\[ dr = dx / \lambda \]

to get

\[ \left( \frac{x}{\lambda} \right)^2 \frac{d^2 R}{d(x/\lambda)^2} + \frac{x}{\lambda} \frac{dR}{d(x/\lambda)} + \left( \frac{x}{\lambda} \right) \lambda^2 R = 0 \]

\[ x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + x^2 R = 0 \]

This is Bessel equation of order 0 and the solution is

\[ R = a J_0(x) + B Y_0(x) \quad \text{or} \]
\[ R = a J_0(\lambda r) + B Y_0(\lambda r) \]

\[ T = (r, z) = \left( A e^{-\lambda z} + B e^{+\lambda z} \right) \left( C J_0(\lambda r) + D Y_0(\lambda r) \right) \]

\[ T(0, z) \text{ finite} \Rightarrow D = 0 \text{ because } Y_0(0) \rightarrow \infty \]
\[ T(r, z \rightarrow \infty) \text{ finite} \Rightarrow B = 0 \]
\[ T(r, z) = A' e^{-\lambda z} J_0(\lambda r) \]
\[ T(r, z) = 0 \Rightarrow J_0(\lambda) = 0 \]

Let \( \lambda_n \) be the nth root of \( J_0 \)

then

\[ T(r, z) = \sum_{n=1}^{\infty} A_n' e^{-\lambda_n z} J_0(\lambda_n r) \]
\[ T(r, 0) = f(r) = \sum_{n=1}^{\infty} A_n' J_0(\lambda_n r) \]
to solve for the $A'_n$, we use the fact that the $J_0(\lambda_n r)$ is orthogonal. Multiply both sides by $r J_0(\lambda_m r) dr$ and integrate from 0 to 1:

$$\int_0^1 f(r) J_0(\lambda_m r) r dr = \sum_{n=1}^{\infty} A'_n \int_0^1 J_0(\lambda_n r) J_0(\lambda_m r) r dr$$

$$= \sum_{n=1}^{\infty} \frac{A'_n}{2} J_1^2(\lambda_m) \delta_{mn}$$

$$= \frac{A'_m}{2} J_1^2(\lambda_m)$$ by the properties of Bessel Functions

and the solution is

$$T(r, z) = 2 \sum_{n=1}^{\infty} \int_0^1 f(r) J_0(\lambda_n r) r dr \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)} e^{-\lambda_n z}$$

The first four values of $\lambda_n$ are 2.404, 5.520, 8.654, 11.792.