S.S. Heat conduction with source term:

Try spherical geometry using a shell balance:

\[ Q_r \bigg|_r - Q_r \bigg|_{r+\Delta r} + S_r 4\pi r^2 \Delta r \bigg|_r = 0 \]

Remember, we can substitute Fourier's Laws anytime to get:

\[ Q_r = -k A \frac{dT}{dr} \]
\[ A = 4\pi r^2 \]

\[-k \left( 4\pi r^2 \frac{dT}{dr} \right) \bigg|_r + k \left( 4\pi r^2 \frac{dT}{dr} \right) \bigg|_{r+\Delta r} + S_r \left( 4\pi r^2 \Delta r \right) \bigg|_r = 0, \]
rewriting we get:

\[ \lim_{\Delta r \to 0} \frac{\left( r^2 \frac{dT}{dr} \right) \bigg|_{r+\Delta r} - \left( r^2 \frac{dT}{dr} \right) \bigg|_r}{\Delta r} = \frac{S_r r^2}{k} \]

or

\[ \frac{d}{dr} \left( r^2 \frac{\partial T}{\partial r} \right) = \frac{S_r r^2}{k} \]

Integrating once we get

\[ r^2 \frac{dT}{dr} = -\frac{S_r r^3}{3k} + C_1 \]

or

\[ \frac{dT}{dr} = -\frac{r S_r}{3k} + \frac{C_1}{r^2} \]

We can use the obvious boundary condition that nothing real goes to \( \infty \):

as \( r \to 0 \) \( \frac{dT}{dr} \to \infty \) unless \( C_1 = 0 \)

\[ \frac{dT}{dr} = -\frac{r S_r}{3k} \]

Alternative: We can also say that the center of a sphere is a point of symmetry, so

\[ \frac{\partial T}{\partial r} \bigg|_{r=0} = 0; \]
we get the same result.
Integrating again
\[ T(r) = -\frac{r^2 S_r}{6k} + C_2 \]

If the surface temperature is \( T_s \) then
\[ T_s = T(R) = -\frac{R^2 S_r}{6k} + C_2 \]

\[ T_s + \frac{R^2 S_r}{6k} = C_2 \quad \text{and rearranging} \]
\[ T(r) = T_s + \frac{R^2 S_r}{6k} \left(1 - \left(\frac{r}{R}\right)^2\right) \]

The flux leaving the sphere \( Q_r = -kA \frac{dT}{dr} \) is
\[ = -4\pi R^2 k \left[ -\frac{RS_r}{3k} \right] \]
\[ = 4\pi R^3 S_r \quad \text{or simply the total heat generated by the sphere!} \]
which makes a lot of sense.
Conduction with Convection

Example 2 **Heat Conduction in a Cooling Fin:**
How to simplify a multi-dimensional problem with judicious approximations.

**True Situation**
1) $T$ is a function of $x,y,z$
2) A small amount of heat is lost from edges
3) Heat transfer coefficient is a function of temperature, position

**Model**
1) For $2B \ll L,W$
   \[ T = T(z) \text{ only} \]
2) No heat is lost from edges
   \[ 2LW \gg (2BW + 4BL) \]
3) Heat flux from fin is given by Newton's Law of cooling
   \[ q = h(T - T_\infty) \]
   \[ h = \text{constant} \]
   \[ T = T(z) \text{ only} \]

Starting from these assumptions, we can model the fin as a 1-D problem:

A balance across an area of a segment $\Delta z$ of the fin gives

Input by conduction $-$ (Output by conduction + Output by convection) = 0

\[ Q_z \mid_{z} - Q_z \mid_{z+\Delta z} - h \ 2 \ W \Delta z (T - T_\infty) = 0 \]
Inserting Fourier's Law: \( Q_z = -k A \frac{dT}{dz} = -k_2 BW \frac{dT}{dz} \). Inserting this into the previous equation:

\[
-k_2 BW \frac{dT}{dz} \bigg|_z + k_2 BW \frac{dT}{dz} \bigg|_{z+\Delta z} - h_2 BW \Delta z (T - T_\infty) = 0
\]

dividing by 2 WΔz KB and taking limit as Δz → 0

\[
\frac{-dT}{dz} \bigg|_z + \frac{dT}{dz} \bigg|_{z+\Delta z} - \frac{h}{kB}(T - T_\infty) = 0
\]

\[
\frac{d^2T}{dz^2} - \frac{h}{kB}(T - T_\infty) = 0
\]

and the boundary conditions:

\[
z = 0 \quad T = T_w \quad \text{— wall temperature}
\]
\[
z = L \quad \frac{dT}{dz} = 0 \quad \text{— no heat flows out of fin end}
\]

Introduce dimensionless variables:

\[
t = \frac{T - T_\infty}{T_w - T_\infty}; \quad \eta = \frac{z}{L}; \quad z = L\eta;
\]

\[
(T_w = T_\infty) dt = dT \quad N = \sqrt[2]{\frac{hL^2}{kB}}
\]

Make dimensionless to get:

1) important groups of parameters
2) bounds on variables
3) **Remember this**

Writing the equation in terms of these variables

\[
\frac{d^2T}{dz^2} + \frac{h}{kB}(T - T_\infty)
\]

\[
\frac{d^2 \left[ (T_w - T_\infty) t \right]}{d \left( L\eta \right)^2} = \frac{h}{kB} \left( T_w - T_\infty \right) t
\]

\[
\frac{d^2 t}{d\eta^2} = \frac{L^2 h}{kB} t = N^2 t
\]

The solutions of the above equation are

\[
t = C_1 \cosh N\eta + C_2 \sinh N\eta \quad \textbf{Remember this}
\]

Now, we need to evaluate \( C_1 \) and \( C_2 \) using the boundary conditions:
\[ t(\eta = 0) = 1 \quad \frac{dt}{d\eta} \bigg|_{\eta = 0} = 0 \]

\[ \frac{dt}{d\eta} = NC_1 \sinh N\eta + NC_2 \cosh N\eta \]

evaluated at \( \eta = 1 \)

\[ 0 = NC_1 \sinh N + NC_2 \cosh N \]

\[ C_2 = -C_1 \frac{\sinh N}{\cosh N} = -C_1 \tanh N \]

\[ t(\eta) = C_1 [\cosh N\eta - \tanh N \sinh N\eta] \]

at \( \eta = 0 \quad t(\eta) = 1 \)

\[ t(0) = 1 = C_1 \left[ \cosh(0) - \tanh(N) \sinh(0) \right] \quad \text{(recall that } \cosh 0 = 1, \sinh 0 = 0) \]

\[ 1 = C_1 \]

\[ t(\eta) = \cosh(N\eta) - \tanh(N) \sinh(N\eta) \]

which can be rearranged to give

\[ t(\eta) = \frac{\cosh N(1-\eta)}{\cosh N} \]

The “fin effectiveness” is defined by

\[ \gamma = \frac{\text{actual heat lost}}{\text{heat lost if fin at } T_w} \]

\[ \gamma = \frac{\int \int h(T(z) - T_\infty) dz dy}{\int \int h(T_w - T_\infty) dz dy} \]

\[ \frac{\int t(z) d\eta}{\int d\eta} = \left( \int_0^w \text{integrals cancel} \right) \]

\[ \int_0^1 \cosh N\eta - \sinh N\eta \tanh N \]

\[ = \frac{1}{N} \left[ \sinh N\eta - \tanh N \cosh N\eta \right]_0^1 = \tanh \frac{N}{N} \]

So what makes for an effective fin?
Look at \( N = \left( \frac{L^2h}{kB} \right)^{1/2} \)

To maximize \( \gamma \) :

\[
0 = \frac{\partial \gamma}{\partial N} = \frac{1 - \tanh^2 N}{N} - \frac{\tanh N}{N^2}
\]

\[
0 = \frac{N - N \tanh^2 N - \tanh N}{N^2}
\]

\( N \neq 0 \) so

\[ N = N \tanh^2 N + \tanh N \]

or

\[ 1 = \tanh^2 N + \frac{\tanh N}{N} \]

The only choices are \( N = 0, \ N = \infty \) for solutions.

\( N = 0 \) is most efficient fin

So, straight fins of constant cross section on a straight wall aren’t so good.

We could see this more easily by looking at \( \frac{\tanh N}{N} \) as \( N \to 0, \infty \)

Both numerator and denominator go to zero, to see if there is a limit.

Using L'Hopital's rule

\[
\lim_{N \to 0} \frac{\tanh N}{N} = \lim_{N \to 0} \frac{d}{dN} \left( \frac{\tanh N}{N} \right) = \frac{1 - \tanh^2 N}{1} = 1
\]

and we see that the efficiency is 1 at \( N = 0 \) and cannot be higher. \( N \to \infty \) means \( \eta \to 0 \).

This means many small fins are better than one or two big fins!

Under the constraint of constant fin weight, what is the optimal length / thickness ratios of a rectangular fin?

\[ Q = \frac{mk(T_w - T_s)}{\rho L^2} N \tanh N \] from analysis of the efficiency of fin

Mass of fin is: \( m = 2 \rho BLW \)

And we can write \( N \) as \( N^2 = \frac{h L^2}{k B} \) or

\[ N = \frac{m}{2W \rho} \left( \frac{h}{k} \right)^{1/2} B^{-3/2} \]

So that we can eliminate \( L \) as a function of \( B \)
and $Q$ becomes

$$Q = C \, N^{-1/3} \tanh N$$

$$\frac{\partial Q}{\partial N} = 0 = -1/3 \, N^{-4/3} \tanh N + \frac{1}{N^{1/3} \cosh^2 N}$$

From trial and error

$$N_{\text{opt}} = 1.419$$ or

$$\left( \frac{L}{B} \right)_{\text{opt}} = 1.419 \left( \frac{h}{kB} \right)^{1/2}$$

Fins: Different geometries

There are a multitude of shapes for fins. Your text has the effectiveness factors, $\eta$, for a variety of configurations. For finned surfaces, the total rate of heat loss is

$$Q = h \Delta T \left[ A_0' + \eta A_f \right]$$

$A_0$ = original area

$A_f$ = fin area

$\eta$ = effectiveness from tables

$A_0' = A_0 + A_f \text{base}$

Water and air are separated by a steel sheet (radiator?) of .125 in thickness. We need to increase heat transfer by adding straight triangular fins of .95 inch base thickness ($t$), 1 in long ($L$) along the sheet with 0.5 in center to center spacing. The sheet is very side.

Water and air are separated by a steel sheet (radiator?) of .125 in thickness. We need to increase heat transfer by adding straight triangular fins of .95 inch base thickness ($t$), 1 in long ($L$) along the sheet with 0.5 in center to center spacing. The sheet is very side.

What side do we put the fins on to maximize heat transfer? How much do we increase $Q$?

To determine which side to put the fins on we examine a plain sheet

and realize that we have 3 resistances in Sevier -

$$H_2O \rightarrow \text{sheet by convection} \rightarrow \text{sheet by conduction} \rightarrow \text{air by convection}$$
\[ Q = \frac{\Delta T}{\sum R} \]
\[ R_{\text{conduction}} = \frac{t_{\text{conduction}}}{K_{\text{sheet}}} \]
\[ R_{\text{convection}} = \frac{1}{h_i} \]

\[ \sum R \leftarrow \text{(below) like electrical resistors} \]
\[ \sum R = \frac{t_{\text{sheet}}}{k_{\text{sheet}}} + \frac{1}{h_{\text{air}}} + \frac{1}{h_{\text{water}}} \]
\[ = \left( \frac{0.125 \text{ in}}{26 \text{ Btu/hr ft}^\circ \text{F}} \right) + \frac{1}{2 \text{ Btu/hr ft}^\circ \text{F}} + \frac{1}{45 \text{ Btu/hr ft}^\circ \text{F}} \]
\[ = (0.004 + 0.5 + 0.022) \text{ hr ft}^\circ \text{F/Btu} \]
\[ \sum R = 0.522 \]

Now we can examine where the major resistance is:

\[ \frac{R_{\text{air}}}{\sum R} = \frac{0.5}{0.522} = 0.96 \]
\[ \frac{R_{\text{sheet}}}{\sum R} = \frac{0.004}{0.522} \approx 0 \]
\[ \frac{R_{\text{H}_2\text{O}}}{\sum R} = \frac{0.022}{0.522} \approx 0.04 \]

So almost all resistance is on air side

none in sheet

little in H\textsubscript{2}O

In cases like this, we want to see where the fins will do the most good. Since the air provides the most resistance, it makes sense to try to decrease this by increasing the area by adding fins to this side. Fins added to the water side will not do too much!

To show this, look at 1 square foot of sheet. At ½ in spacing, this means 24 fins. \( A_0 = 1 \text{ ft} \), but what is \( A_f \)? \( A'_f \)

\[ 0.05 \text{ in} \times 1 \text{ ft is the base Area} \]
\[ A'_f = 1 \text{ ft}^2 - 24 \text{ fins} \left( \frac{0.05 \text{ in} \cdot 1 \text{ ft}}{12 \text{ in/ft fin}} \right) = 9 \text{ ft}^2 \]
\[ A_f = (24 \text{ fins}) (1 \text{ ft width}) (2 \times 1/12 \text{ ft}) = 4 \text{ ft}^2 \]

\[ \uparrow \text{ top & bottom} \]
\[ \uparrow \text{ Length} \]
Neglect cosine of angle since fins are long and narrow. Calculate $\eta$ for fins from value of $N$ - tables for $\eta$

$$L_c = L + \frac{t}{2} = 1.025 \text{ inches} = 0.85 \text{ feet}$$

$$A_m = \frac{t L_c}{2} = (0.025)(1.025) = 0.0256 \text{ in}^2 = 1.78 \times 10^{-4} \text{ ft}^2$$

Fins are steel with $k = 26 \text{ Btu/hr ft } ^\circ\text{F}$

$$\begin{align*}
\text{Air side } & h = 2 \\
\text{Water side } & h = 45
\end{align*}$$

Tables in handbooks

$$N = L_c^{3/2} \left( \frac{k}{k A_m} \right)^{1/2} = 2.44$$

From homework problem 10.2, the solution $\eta$ we get

$$\begin{align*}
\eta_{\text{air}} &= 0.89 \\
\eta_{\text{water}} &= 0.39
\end{align*}$$

Now to compare the heat flows –

No fin baseline

$$Q_0 = \frac{A_0 \Delta T}{\frac{1}{2} + \frac{1}{45}} = 1.91 A_0 \Delta T$$

#1 no fins

#2 with fins, we have

$$Q_f = h_1 A_0 \left( T_1 - T_w \right) - \text{no fins side}$$

$$Q_f = h_2 A_f \left( T_w - T_2 \right) \eta + h_2 A'_0 \left( T_w - T_2 \right)$$

$$A'_0 = A_0 - A_{fb}$$

We don’t know $T_w$ in this case so:

$$q_1 = q_2$$

$$h_1 A_0 \left( T_1 - T_w \right) = h_2 \left( T_w - T_2 \right) \left( A_f \eta + A'_0 \right)$$

Solving for $T_w$ gives

$$T_w = \frac{h_1 T_1 A_0 + h_2 T_2 \left( A_f \eta + A'_0 \right)}{h_1 A_0 + h_2 \left( A_f \eta + A'_0 \right)}$$

And $Q_f$ is

$$Q_f = h_1 A_0 \left( T_1 - T_w \right)$$
\( Q_f = \frac{A_0 (T_1 - T_2)}{1 + \frac{A_0}{h_2 (A'_0 + A_f \eta)} + \frac{1}{h_f}} \)

Enhancement due to change in area or decrease in \( R \) or

\[ \frac{Q_f}{A_0\Delta T} = \frac{1}{1 + \frac{A_0}{h_2 (A'_0 + A_f \eta)} + \frac{1}{h_f}} \]

to get to

Air side with fins

\[ \frac{Q_f}{A_0\Delta T} = 7.4 \]

Water side

\[ \frac{Q_f}{A_0\Delta T} = 1.96 \]

Relative change

\[ Q_f - Q_0 = (7.4 - 1.91) A_0\Delta T \]
\[ Q_f / Q_0 = 3.87 \]

So putting fins on the side of the greatest resistance is best. On the least resistance side, it doesn’t help at all.

Air

\[ Q_f - Q_0 = (7.4 - 1.91) A_0\Delta T \]
\[ Q_f / Q_0 = 3.87 \]

Water

\[ Q_f - Q_0 = (1.96 - 1.91) A_0\Delta T \]
\[ Q_f / Q_0 = 1.03 \]

Note that \( Q_f / Q_0 \) is not \( \frac{A_0 + A_f \sim 5}{A_0} \), but less because of \( \gamma \) and resistance on air side.

We want

\[ \frac{1}{h_f} \left[ \frac{A_0}{A'_0 + \gamma A_f} \right] \gg \frac{1}{h_{\text{non-fin}}} \]

To get near linear improvement

\[ \frac{1}{2} \left[ \frac{1}{5} \right] \gg \frac{1}{45} \]
\[ \frac{1}{10} \gg \frac{1}{45} \] Check!