THE LAPLACE TRANSFORM

1. Origin of the LaPlace Transform.
   A periodic function $f(x)$ on the interval $-\pi < x < \pi$ can be expanded in a Fourier series
   
   $$ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) $$

   where
   
   $$ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx $$
   $$ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx $$

   Since
   
   $$ \cos nx = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}), $$

   we can write the Fourier series as
   
   $$ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} $$

   where we define
   
   $$ c_n = \begin{cases} 
   \frac{1}{2}(a_n + b_n i) & \text{for } n \geq 0 \\
   \frac{1}{2}(a_{-n} + i b_{-n}) & \text{for } n < 0 
   \end{cases} $$

   Then
   
   $$ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx $$

   If we wish to represent a function $f(x)$ on an interval $-L \leq x \leq L$ as a Fourier series, we simply introduce the new variable
   
   $$ x' = \frac{\pi}{L} x \quad \text{Then we can write} $$
   
   $$ f \left( \frac{L}{\pi} x' \right) = \sum_{n=-\infty}^{\infty} c_n e^{-inx'} $$
where
\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right) e^{-inx/L} \, dx' \]

and returning to the variable \(x\) we have:
\[ f(x) = \sum_{-\infty}^{\infty} c_n^{(L)} e^{-inx/L} \tag{1} \]

where
\[ c_n^{(L)} = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-inx/L} \, dx \tag{2} \]

Thus the function \(f(x)\) defines uniquely the coefficients \(c_n^{(L)}\) on the interval \((-L,L)\) and vice versa.

Now our objective is to expand an aperiodic function in a Fourier series, so we let \(L \to \infty\); in other words we regard the case of an aperiodic function as that of a periodic function with infinite period.

If we fix \(n\), \(n/L \to 0\) as \(L \to \infty\) and hence from equation (2)
\[ \lim_{L \to \infty} 2L c_n^{(L)} = \lim_{L \to \infty} \int_{-L}^{L} f(x) e^{-inx/L} \, dx = \int_{-\infty}^{\infty} f(x) dx \]

This limit is a single constant and hence cannot determine \(f(x)\).

We observe however that as \(L \to \infty\) the set of numbers of the form \(n\pi/2L\) with \(n = 0, \pm 1, \pm 2, ...\) becomes more and more dense on the real line. This motivates us to replace the quantity \(n\pi/L\) by a continuous variable \(\omega\), and to keep \(\omega\) fixed as \(L \to \infty\).

We obtain the limiting function
\[ \hat{f}(\omega) = \lim_{L \to \infty} 2L c_n^{(L)} \text{ or } \hat{f}(\omega) = \int_{-\omega}^{\omega} f(x) \, dx \]
when \(L \to \infty\) the summation of equation (1) can be replaced by an integral. In doing so, we observe that this is a summation w.r.t \(n\) which can be written as \((\Delta n = 1)\):
\[ \sum_{-\infty}^{\infty} c_n^{(L)} e^{-inx/L} \Delta n = \sum_{-\infty}^{\infty} 2L c_n^{(L)} e^{-inx/L} \frac{\pi}{2\pi} \Delta n = \frac{1}{2\pi} \sum_{-\infty}^{\infty} 2L c_n^{(L)} e^{-inx/L} \frac{\pi}{L} \Delta n = \frac{1}{2\pi} \sum_{-\infty}^{\infty} 2L c_n^{(L)} e^{-i\omega} \Delta \omega \]

Therefore, when \(L \to \infty\), \(\pi n/L\) can be considered as a continuous variable \(\omega\) and the above summation can be substituted by an integral:
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2L c_n^{(L)} e^{-i\omega} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega} \, d\omega \]

The two relations
\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega} \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega} \, d\omega \tag{3} \]
form a Fourier transform pair. They are equivalent to:
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} \, d\omega
\]

These relations are very useful in dealing with problems for which \( x \) is on the real line \( (-\infty < x < \infty) \). The trouble is that the Fourier transform of a function \( f(x) \) exists only if the function is absolutely integrable:
\[
\int_{-\infty}^{\infty} |f(x)| \, dx < \infty
\]

Unfortunately, many of the functions in which we shall be interested do not meet this requirement, typical examples being the unit step function and the sine function. It is convenient to modify in the form (consider now functions that are nonzero for \( x \geq 0 \)):
\[
\int_{-\infty}^{\infty} |f(x)| e^{-ax} \, dx < \infty
\]

so that there exists a constant \( a \geq 0 \) such that (6) holds. The smallest \( a \) for which (6) holds is called the abscissa for convergence and is denoted by \( a_0 \).

If (5) holds, then we can take the Fourier transform of \( e^{-ax} f(x) \) which for \( a > a_0 \) is guaranteed to exist. This Fourier transform can be written as (remember that we assumed \( f(x) = 0, x < 0 \)):
\[
\hat{f}(a+\omega) \int_{-\infty}^{\infty} f(x) e^{-(a+i\omega)x} \, dx
\]

Defining the complex frequency \( s \) by \( s = a + i\omega \) we obtain the Fourier transform of \( e^{-ax} f(x) \) which now is called the LaPlace transform of \( f(x) \):
\[
\hat{f}(s) = \int_{0}^{\infty} f(x) e^{-sx} \, dx
\]

Therefore the LaPlace transform is a natural result of providing the Fourier transform with a built-in convergence factor

2. General Properties of the LaPlace Transform

We shall be looking at the LaPlace transform (LT) of a function

\[
F(t) \text{ defined for } t \geq 0 : \quad \begin{cases}
    f(s) = \int_{0}^{\infty} F(t) e^{-st} \, dt \\
    F(t) = \frac{1}{2\pi i} \int_{0}^{\infty} f(s) e^{st} \, ds
\end{cases}
\]

LT pair
By capital letter we denote the function and by lower case letters its transform. Note that the latter is a function of $s$.

(a) **LaPlace Transform is a linear operation:**

\[ F_1(t) + F_2(t) \xrightarrow{\mathcal{L}} f_1(s) + f_2(s), \quad a F(t) \xrightarrow{\mathcal{L}} a f(s) \]

\[ F_1(t) + F_2(t) \xrightarrow{\mathcal{L}^{-1}} f_1(s) + f_2(s), \quad a F(t) \xrightarrow{\mathcal{L}^{-1}} a f(s) \]

(b) **LaPlace Transform of derivatives:**

\[ L\left\{ \frac{dF}{dt} = \frac{dF}{dt} = e^{-st} \frac{dF}{dt} dt = e^{-st} F(t) \right\}_0^\infty + s \int_0^\infty e^{-st} F(t) dt \]

\[ = sf(s) - F(0) \]

\[ L\left\{ \frac{d^2F}{dt^2} \right\} = s^2 f(s) - sF(0) - \dot{F}(0) \]

\[ L\left\{ F^{(n)}(t) \right\} = s^n f(s) - s^{n-1} F(0) - \cdots - F^{(n-1)}(0) \]

For a function of more than one variable: $F(x, t)$ we have:

\[ L \left\{ \frac{\partial F}{\partial t} \right\} = sf(x, s) - F(x, 0) \]

\[ L \left\{ \frac{\partial F}{\partial x} \right\} = \frac{\partial f(x, s)}{\partial x} : \text{no effect since the transform is with respect to } t \]

Example: Heat equation:

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \xrightarrow{\mathcal{L}} s\bar{T}(x, s) - T_0 = \alpha \frac{d^2 \bar{T}(x, s)}{dx^2} \]

\[ T(x, 0) = T_0 \]

\[ T(0, t) = T_1 \]

\[ T(1, t) = T_2 \]

\[ \left\{ \begin{array}{c}
T(0, s) = T_1 / s \\
\bar{T}(1, s) = T_2 / s
\end{array} \right. \xrightarrow{\mathcal{L}} \]

The initial condition is automatically incorporated when we take the LaPlace transform of the differential equation. Taking LaPlace transform reduces the partial differential equation to a second order ODE.
(c) **Initial and Final Value Theorems:**

Initial value theorem: \( \lim_{s \to 0} s f(s) = \lim_{t \to 0} F(t) \)

Final value theorem: \( \lim_{s \to 0} s f(s) = \lim_{t \to \infty} F(t) \)

(d) **LaPlace Transform of an Integral:**

\[
\int_{0}^{t} F(t) \, dt \xrightarrow{LT} \frac{f(s)}{s}
\]

(e) **Differentiation and Integration of** \( f(s) \):

By suitable differentiation and integration, it is possible to show:

\[
\frac{df(s)}{ds} = L[-tF(t)], \quad \int_{0}^{\infty} f(s) \, ds = L\left[\frac{F(t)}{t}\right]
\]

(f) **The shifting or translation theorem:**

\[
L\left[e^{at}F(t)\right] = f(s-a)
\]

(g) **Periodic functions:**

For a periodic function, \( F(t) \) with period \( \tau \)

\[
L\left[F(t)\right] = \int_{0}^{\tau} e^{-st}F(t) \, dt \quad \frac{1}{1-e^{-\tau s}}
\]

(h) **The Convolution integral:**

Perhaps one of the most general and important properties of the LaPlace Transform relates to the convolution integral of two different time functions \( F_1(t) \) and \( F_2(t) \).

Thus, if:

\[
F_1(t) \xrightarrow{LT} f_1(s), \quad F_2(t) \xrightarrow{LT} f_2(s) \quad \text{then}
\]

\[
f_1(s) f_2(s) = L\left[\int_{0}^{t} F_1(\xi) F_2(t-\xi) \, d\xi\right], \quad \xi \text{ variable of integration}
\]

or, from simple symmetry:

\[
f_1(s) f_2(s) = L\left[\int_{0}^{t} F_2(\xi) F_1(t-\xi) \, d\xi\right]
\]
Note that if the inverse LaPlace Transforms of two functions \( f_1(s) \) and \( f_2(s) \) are known, the inverse LT of their product can be obtained by an integration.

(i) Dirac delta function

\[
\delta(t-t_1) = \lim_{t \to t_1^+} \{ U(t-t_1) - U(t-t_1-\tau) \} \\
U(t-t_1): \text{until step function}
\]

Some Simple Transforms:

\[
F(t) = \frac{e^{-at}}{e^{-st}} \\
\frac{f(s)}{s+a} \\
\sin at = \frac{a}{s^2 + a^2} \\
\begin{cases} 
0 < t < 0 \\
a & t > 0 \\
e^{-bt} \sin at = \frac{a}{(s+b)^2 + a^2}
\end{cases}
\]


The LT of piecewise continuous functions like:
One way is by the definition of the LaPlace Transform:

\[
L[H_1(t)] = \int_0^\infty H_1(t)e^{-st} \, dt = \int_0^3 4e^{-st} \, dt + \int_3^\infty 2e^{-st} \, dt
\]

\[
= \frac{4}{s} e^{-st} \left[ \frac{1}{s} e^{-st} \right]_0^3 + \lim_{R \to \infty} \int_0^R 2e^{-st} \, dt
\]

\[
= \frac{4}{s} - \frac{2}{s} e^{-3s}
\]

\[
L[H_3(t)] = \int_0^\infty H_3(t)e^{-st} \, dt = \int_0^1 0 \, dt + \int_1^2 te^{-st} \, dt + \int_2^\infty e^{-st} \, dt
\]

\[
= \cdots + \frac{s+1}{s^3} \left( e^{-s} - e^{-2s} \right)
\]

Another way is to express \(H_1, H_2, H_3\) as sums of functions for which we know their transforms. Then we can avoid tedious integration.

Note:

If \(G(t) = \begin{cases} 0 & 0 < t < a \\ F(t-a) & t > a \end{cases}\) ← Note \(T(t-a)\) form

← But LaPlace transform uses \(F(t)\)

then \(L[G(t)] = e^{-as} F(s)\)

Example:

\[
H(t) = \begin{cases} 4 & 0 < t < 3 \\ 2 & t > 3 \end{cases} = 4 + \begin{cases} 0 & 0 < t < 2 \\ -2 & 3 > t \end{cases}
\]

← Call this \(G(t)\)

\[
\text{Note that } F(t-3) = -2
\]

Therefore, \(L[H_1(t)] = L[4] + L[G(t)] = \frac{4}{s} + e^{-3s} L[-2] = \frac{4}{s} - \frac{2}{s} e^{-3s}\)

\[
H_2(t) = \begin{cases} 3 & 0 < t < 2 \\ s & 2 > t \end{cases} = 3 + \begin{cases} 0 & 0 < t < 2 \\ 2 & 2 > t \end{cases}
\]
In order to be able to use the note of the previous page, F1 must be a function of t−1 and F2 a function of t−2.

\[ t = (t-1)+1, \text{ so define } F_1(t) = t+1, \text{ then } F_1(t-1) = t+1-1 = t \]
\[ 1-t = -(t-1) = -(t-2)-1, \text{ so define } F_2(t) = -t-1, \text{ then } F_2(t-2) = -(t-2)-1 = -t+1 \]

Then

\[
H_3(t) = \begin{cases} 
0 & 0 < t < 1 \\
F_1(t-1) & 1 < t < 2 \\
F_2(t-2) & 2 < t < \infty \\
1-t & 2 < t < \infty 
\end{cases}
\]

\[
L[H_3(t)] = L[t+1]e^{-s} + e^{-2s}L[-t-1] = e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right] - e^{-2s}\left[\frac{1}{s} + \frac{1}{s^2}\right] = L\left[e^{-s} - e^{-2s}\right] \left(\frac{s+1}{s^2}\right)
\]

Find the Inverse Transform of \( f(s) \) \( g(s) \) using the Convolution Theorem.

The method:

1. Find the inverse transform of \( f(s) \) and \( g(s) \)

\[
L^{-1}\left[f(s)\right] = F(t) \\
L^{-1}\left[g(s)\right] = G(t)
\]

2. Set up the integral \( \int_0^t F(t)G(t-\tau)d\tau \)

3. Integrate with respect to \( \tau \).

Example:

\[
L^{-1}\left[\frac{1}{s(s-a)}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s-a}\right] = \frac{1}{a}e^{at}
\]

\[
L^{-1}\left[\frac{1}{s}\right] = 1, \quad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}
\]

\[
L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s-a}\right] = \int_0^t e^{-a(t-\tau)}d\tau = \frac{1}{a}e^{at}
\]

Example:

\[
L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{1}{s^2+1}\right]
\]

\[
L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t \rightarrow L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = \int_0^t \sin \tau \cdot \sin (t-\tau) d\tau = \frac{1}{2} \sin t \cdot \frac{1}{2} \cos t
\]
Partial Fractions:

How to convert $f(x)/g(x)$ to partial fractions if $f(x)$ and $g(x)$ are polynomials.

The degree of the numerator $f(x)$ must be less than the degree of the denominator.

If the degree of $f(x)$ is greater than the degree of $g(x)$ we long division to reduce the problem:

Example: \[
\frac{x^3 + 2x}{x^2 - x + 1} = (x+1) + \frac{2x-1}{x^2 - x + 1}
\]

If the degree of $f(x)$ is equal to the degree of $g(x)$ you can add and subtract terms in the numerator.

Example:

\[
\frac{x^2 - x - 1}{x^2 - 2x + 3} = \frac{x^2 - 2x + 3}{x^2 - 2x + 3} - \frac{x - 1}{x^2 - 2x + 3} = \frac{3x + 4}{x^2 - 2x + 3} - \frac{1}{3(3x+2)}
\]

Now suppose the degree of $f(x)$ is less than the degree of $g(x)$. We want to express $f(x)/g(x)$ as a sum of partial fractions:

Factor the denominator as much as possible: you end up with terms of the form $(px+q)$ or $(ax^2 + bx + c)$ where $b^2 - 4ac < 0$ so the roots of the quadratic are complex.

Rule 1: If $(px+q)$ occurs only once it corresponds to a factor of the form $A/(px+q)$ on the right side of

\[
\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}
\]

Rule 2: If $(ax^2 + bx + c)$ occurs only once as a factor of $g(x)$ it corresponds to a partial fraction of the form $\frac{A_1x+B_1}{(ax^2 + bx + c)}$.

If $(ax^2 + bx + c)^m$ occurs, include:

\[
\frac{A_1x+B_1}{(ax^2 + bx + c)} + \frac{A_2x+B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2 + bx + c)^m}
\]

Once you have set up the right side put everything under a common denominator and compare the resulting numerator to $f(x)$. 

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Method A: If a factor \((pcx + q)(pcx + q)\) occurs only once, let \(x = -q/p\). This is a simple way of finding the corresponding constants.

Method B: Group the coefficients of \(x, x^2, x^3\ldots\) and constant terms. The coefficient of \(x^k\) in \(f(x)\) must equal the coefficient on the right. Solve the simultaneous equations.

Example: \[
\frac{11x + 2}{2x^2 - 5x - 3} = \frac{11x + 2}{(2x + 1)(x - 3)} = \frac{A}{2x + 1} + \frac{B}{x - 3}
\]

Put under common denominator: \[
\frac{11x + 2}{2x^2 - 5x - 3} = \frac{A(x - 3) + B(2x + 1)}{2x^2 - 5x - 3}
\]

True if and only if: \(11x + 2 = A(x - 3) + B(2x + 1)\)

Method A: Set \(x = 3 \rightarrow 7B = 35 \rightarrow B = 5\)
Set \(x = -1/2 \rightarrow -7/2A = -7/2 \rightarrow A = 1\)

Method B: Group powers of \(x\):
\(11x + 2 = (A + 2B)x + (-3A + B) \therefore A + 2B = 11, -3A + B = 2 \rightarrow A = 1, B = 5.\)
Linearity and Superposition

The symbol \( \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} \)
describes a set of manipulations (differentiation, multiplication and subtraction which assigns
to any sufficiently differentiable function \( u(x,t) \) a new function of \( x \) and \( t. \) Such an assignment
(or transformation) of one function to another function is call an operator. Since the basic
process is partial differentiation we call the particular operator which assigns to each \( u \) the
function.

\[ \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} \]
a partial differential operator. For brevity, we call such an operator by \( L \) and the function it
assign to a particular \( u \) by \( L[u] \).

The above operator \( L \) has a very special property. We note that if \( u(x,t) \) and \( v(x,t) \) are any
twice differential functions, the same is true of a linear combination \( \alpha u + \beta v, \) where \( \alpha, \beta \) are
any constants. Thus \( L[\alpha u + \beta v] \) is defined. By the rules of partial differentiation:

\[ L[\alpha u + \beta v] = \alpha L[u] + \beta L[v] \]

An operator having these properties is called linear operator.

An equation which equates \( L[u] \) to a given function \( F: \)

\[ L[u] = F \]
is called a linear partial differential equation. The heat equation is an example of a linear
partial differential equation (PDE). The heat equation without generation terms, \((F \equiv 0), \) is an
example of an homogeneous linear PDE.

The unknown function \( u(x,t) \) is in general not determined by the differential equation alone.
We must also prescribe boundary and/or initial conditions. The transformation associating
with a function \( u(x,t) \) its initial values \( u(x,0) \) is also a linear operator, which we may denote,
say by \( L_1[u] \) \( s \) \( u(x,0). \) Similarly,

\[ L_2[u] \equiv u(0,t) \]
\[ L_3[u] \equiv u(L,t) \]
are linear operators.

The initial-boundary value problem may thus be written in the form:

\[ L[u] = F(x,t) \]
\[ L_2[u] = f_1(x) \]
\[ L_2[u] = 0 \] (unsteady heat conduction in a slab
\[ L_3[u] = 0 \] with prescribed initial temperature,)
\[ L_3[u] = 0 \]
This is a system of linear equations. We call a system consisting of a linear PDE together with a set of linear subsidiary conditions a linear problem.

Consider a linear problem of the form:

\[
\begin{align*}
L[u] &= F \\
L_1[u] &= f_1 \\
L_2[u] &= f_2 \quad (1) \\
\vdots \\
L_k[u] &= f_k \\
\end{align*}
\]

where the first equation is a linear PDE while the others are linear initial or boundary conditions.

Suppose we can find a particular solution \( v \) of the differential equation

\[
L[v] = F
\]

which need not satisfy any of the other conditions. Then we can define the new dependent variable:

\[
w = u - v
\]

By the linearity of \( L \) we obtain the equation:

\[
L[w] = L[u] - L[v] = 0
\]

That is, \( w \) satisfies a homogeneous differential equation.

We have thus shown that any solution \( u \) of the equation \( L[u] = F \) can be written as the sum of any particular solution \( v \) of this equation and a solution \( w \) of the corresponding homogeneous equation: \( u = v + w \).

If we have a particular solution \( v \) of \( L[v] = F \), we can reduce the problem (1) to a new problem of the same kind, but with \( F = 0 \). For putting \( w = u - v \) we have from the linearity:

\[
\begin{align*}
L[w] &= 0 \\
L_1[w] &= f_1 - L_1[v] \\
L_2[w] &= f_2 - L_2[v] \\
\vdots \\
L_k[u] &= f_k - L_k[v] \\
\end{align*}
\]

The linearity of the problem (1) can also be used to split it into simpler subproblems. Suppose that \( u_0 \) is the solution of:

\[
\begin{align*}
L[u_0] &= F \\
L_1[u_0] &= 0 \\
L_2[u_0] &= 0 \\
\vdots \\
L_k[u_0] &= 0 \\
\end{align*}
\]
that $u_*$ solves

\[
\begin{align*}
    L[u_1] &= 0 \\
    L_1[u_1] &= f_1 \\
    L_2[u_1] &= 0 \\
    &\vdots \\
    L_k[u_1] &= 0,
\end{align*}
\]

with analogous conditions for $u_2, u_3, \ldots u_k$. Each of the functions $u_0, u_1, \ldots, u_k$ involves only one piece of the data $F, t_1, t_2, \ldots, t_k$. By linearity we find that the function: $u = u_0 + u_1 + u_2 + \ldots + u_k$ satisfies (1).