(A) Be able to look at a differential equation and tell things about it.
1) Ordinary differential equation or Partial Differential Equation?
\[ \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} + u \frac{\partial P}{\partial x} \] (1)
is partial because \( P \) depends on both \( x \) and \( t \). However,
\[ p^2 \frac{dP}{dx} + \frac{d^2 P}{dx^2} - x^3 = 0 \] (2)
is ordinary, because only derivatives with respect to \( x \) appear.
2) What order?
\[ \frac{d^2 P}{dx^2} + x = 0 \] (3)
is second-order because the highest derivative is a second derivative, whereas
\[ y^5 \frac{dy}{dx} + \sin y = 0 \] (4)
is first order.
3) Linear or nonlinear?
\[ x^4 \frac{dy}{dx} + 2 \sin(x)y = 5x^4 \] (5)
is linear, because \( y \) and its derivatives never appear in a nonlinear function (e.g. \((y')^2\), \(\sin(y)\), etc.).
\[ yy' + \sin(y) = x \] (6)
is nonlinear both because of the \( yy' \) and because of the \( \sin y \).
4) Homogeneous or inhomogeneous? Easy way to tell – if \( y = 0 \) solves the differential equation, it is homogeneous. Or – are there any terms in the DE that do not contain \( y \) or its derivatives?
\[ y' + 2y = 4x^3 \] (7)
is inhomogeneous – the \( 4x^3 \) term does not depend on \( y \), and so is an inhomogeneous term.
\[ y^3 y'' + \sin(y) \sin(x)y' + 2y = 0 \] (8)
is homogeneous – all terms depend on \( y \), so that plugging in \( y = 0 \) does solve the equation.

B) How to solve them?
First off – you do not know yet how to solve any PDE’s. All we know are ODE’s.

1) Second order ODE’s – you only know how to solve one kind, which are linear, second-order ODEs with constant coefficients. They can be homogeneous or inhomogeneous. Example:
\[ 2y'' + 2y' + 5y = 5x \] (9)
(Note this is inhomogeneous).
First – solve the homogeneous equation, by plugging in a trial solution
\[ y = e^{\lambda x} \] (10)
and solving for the \( \lambda s \) that work. In this case, this gives
\[ 2\lambda^2 + 2\lambda + 5 = 0, \] (11)
which is solved by
\[
\lambda = \frac{-4 \pm \sqrt{4 - 40}}{4} = \frac{-4 \pm \sqrt{-36}}{4} = -1 \pm \frac{3}{2}i
\]  
(12)
So the solution to the homogeneous equation is
\[
y = Ae^{-x+\frac{3x}{2}} + Be^{-x-\frac{3x}{2}}.
\]  
(13)
Note also that the \(i\) bits can also be expressed as sines and cosines:
\[
y = Ae^{-x} \sin \left(\frac{3x}{2}\right) + Be^{-x} \cos \left(\frac{3x}{2}\right).
\]  
(14)
We’ve done the homogeneous equation – now how to find the solution to the inhomogeneous equation? This is called the particular solution. Easiest way – ‘undetermined coefficients’. Basically – try to guess the right answer. Here the inhomogeneous term is \(2x\) – a first order polynomial. So try a particular solution that is also a first order polynomial:
\[
y_p = Rx + S.
\]  
(15)
Plug this in – \(y_p'' = 0, y_p' = R\). Plugging these in, we get
\[
2R + 5Rx + 5S = 5x.
\]  
(16)
We must gather all the “\(x\)” terms and all the “\(1\)” terms (those with no \(x\)’s). So the \(x\) terms are
\[
5R = 5
\]  
(17)
or \(R = 1\). The “\(1\)” terms give
\[
2R + 5S = 0,
\]  
(18)
or \(5S = -2R = -2\), giving \(S = -2/5\).
So the general solution is
\[
y = Ae^{-x} \sin \left(\frac{3x}{2}\right) + Be^{-x} \cos \left(\frac{3x}{2}\right) + x - \frac{2}{5}.
\]  
(19)
Warning: to obtain the general solution to a second-order ODE, you must be sure you have two linearly independent solutions. Otherwise, you really only have one solution. How do you check? Take the Wronskian
\[
W(x) = y_1y_2' - y_2y_1'.
\]  
(20)
If it is zero – then your solutions are linearly dependent. If it is not zero, then they are linearly independent and you’re fine.
This comes up primarily when you get a ‘double root’ when plugging in (10) and solving for \(\lambda\). For example: if you plug \(y(x) = e^{\lambda x}\) into
\[
y'' + 2y' + y = 0,
\]  
(21)
you get
\[
\lambda^2 + 2\lambda + 1 = 0,
\]  
(22)
or \((\lambda + 1)^2 = 0\). If you wrote the general solution
\[
y(x) = Ae^{-x} + Be^{-x}
\]  
(23)
you’d be wrong, because the two solutions are actually the same thing. The wronskian would be zero. In cases like this, you typically try to multiply the second solution by \(x\) - here you’d get
\[
y(x) = Ae^{-x} + Bxe^{-x}
\]  
(24)
and you’d wind up with the right answer. Try it and see. And then try the Wronskian – it will indeed be non-zero, and so the two solutions are linearly independent.

2) FIRST ORDER: you know how to do more first order differential equations. The first thing to check is whether the ODE is linear or not.

If it is linear – you can find an integrating factor. Remember, write your equation in the form

$$y' + P(x)y = Q(x). \quad (25)$$

Then the integrating factor is

$$\mu = e^{\int P(x)dx}, \quad (26)$$

and you multiply the whole equation by this integrating factor. What happens is that the left-hand side can then be re-written as

$$\frac{d}{dx} \left( ye^{\int P(x)dx} \right) = Q(x)e^{\int P(x)dx} \quad (27)$$

and you can then simply integrate both sides to solve the equation. For example:

$$xy' + 2y = 3x \quad (28)$$

is a first-order, linear differential equation that we can solve with integrating factors. First, write it in the desired form – we need the coefficient of $y'$ to be one (not $x$, as it is here) so divide by $x$:

$$y' + \frac{2}{x}y = 3 \quad (29)$$

Now, the integrating factor is

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^\ln x^2 = x^2 \quad (30)$$

We thus have to multiply the ODE by $x^2$, giving

$$x^2y' + 2xy = 3x^2, \quad (31)$$

and notice that the left hand side is simply

$$\frac{d}{dx}(x^2y) = 3x^2. \quad (32)$$

Integrate both sides:

$$yx^2 = x^3 + C, \quad (33)$$

so that the general solution is

$$y(x) = x + \frac{C}{x^2}. \quad (34)$$

If we were given a boundary condition like $y(1) = 2$, then we’d plug it in now:

$$y(1) = 2 = 1 + C, \quad (35)$$

or $C = 1$, meaning the solution would be

$$y(x) = x + \frac{1}{x^2}. \quad (36)$$

3) What if you have a nonlinear first-order PDE? First thing to do is check if it is separable. Can you write it as

$$\frac{dy}{dx} = f(x)g(y)? \quad (37)$$
If so – then separate the two variables like
\[ \frac{dy}{g(y)} = f(x)dx \] (38)
and integrate both sides. For example
\[ \frac{dy}{dx} = \frac{\cos(x)}{3y^2} \] (39)
is separable: write it as
\[ 3y^2dy = \cos(x)dx \] (40)
and integrate:
\[ y^3 = \sin(x) + C \] (41)
which you can simplify to give
\[ y(x) = (C + \sin(x))^{1/3}. \] (42)

4) What if your nonlinear first-order ODE is *not* separable? Then you hope and pray it is ‘exact’. Write your equation in the form
\[ M(x, y)\frac{dy}{dx} + N(x, y)y = 0. \] (43)
Then, if
\[ \frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}, \] (44)
then the ODE is exact. What this means is that you can find a function \( \Phi(x, y) \) such that
\[ M(x, y) = \frac{\partial \Phi}{\partial y} \] (45)
\[ N(x, y) = \frac{\partial \Phi}{\partial x}, \] (46)
and the solution to the equation is
\[ \Phi(x, y) = C \] (47)
where \( C \) is a constant that is determined by satisfying the boundary or initial condition.
For example, look at the generally nasty ODE
\[ \left( \ln(x + y) + \frac{y}{x + y} \right) \frac{dy}{dx} + \frac{y}{x + y} = 0. \] (48)
Is it exact?
\[ \frac{\partial M}{\partial x} = \frac{1}{x + y} - \frac{y}{(x + y)^2} \] (49)
\[ \frac{\partial N}{\partial y} = \frac{1}{x + y} - \frac{y}{(x + y)^2}. \] (50)
Since the two are equal – the equation is exact. So we have to find the function \( \Phi \), which we do by integrating equation (45) or (46). The latter one looks simpler – so let’s try that.
\[ \Phi = \int N(x, y)dx = \int \frac{y}{x + y}dx = y\ln(x + y). \] (51)
To see if it worked – take $\frac{\partial \Phi}{\partial y}$ and see if we get $M(x, y)$:

$$\frac{\partial y \ln(x + y)}{\partial y} = \ln(x + y) + \frac{y}{x + y}$$

which is precisely $M(x, y)$. So – this means the ODE can be written as

$$\frac{d}{dx} (\Phi) = 0,$$

or

$$\Phi(x, y) = y \ln(x + y) = C.$$  

Here you can not solve explicitly for $y(x)$ – you must leave the solution ‘implicit’.

d) – what if your nonlinear, first-order ODE is NOT exact? Well, you can do it but it can be annoying. Do not worry about doing this on an exam – there are lots of ways you can get screwed up, and time can be of the essence. However, I want you to have seen the basic idea, so you know where to look in case you ever do get a problem of this sort.

The idea is to multiply the ODE by an integrating factor $\mu(x, y)$ in order to make the ODE exact. That is, you want to find a $\mu(x, y)$ so that

$$\frac{\partial (\mu(x, y)M(x, y))}{\partial x} = \frac{\partial (\mu(x, y)N(x, y))}{\partial y}.$$  

Laplace Transforms

The Laplace transform – and all transforms, for that matter – take a function and represent it in a different way. Same info, different language. Laplace is particularly useful for solving initial-value linear differential equations.

The Laplace transform is defined as

$$Y(s) = \mathcal{L} [y(t)] = \int_0^\infty y(t)e^{-st}dt$$

You can either calculate it directly, or use the tables. The tables are pretty easy once you get used to them. The trick is to get things into the form you need. Note the Laplace transform is linear – so if you are taking the transform of two terms, you can take the transform of each individually and just add the results. Or if you are taking the transform of 8 times a function, you can just transform the function and multiply by 8. For example, to transform

$$\mathcal{L} \left[ 8t^3 e^{4t} - 4\cos(5t) \right] = 8\mathcal{L} \left[ t^3 e^{4t} \right] - 4\mathcal{L} \left[ \cos(5t) \right].$$

Now, each of these can be simply looked up in the table. $t^3 e^{4t}$ in Table 3.1, $t^n e^{at}$, with $n = 3$ and $a = 4$. And $\cos 5t$, with $a = 5$. From those entries in the table, we get

$$8\mathcal{L} \left[ t^3 e^{4t} \right] = 8 \frac{4!}{(s - 4)^4}$$

$$-4\mathcal{L} \left[ \cos 5t \right] = -4 \frac{s}{s^2 + 25},$$

giving

$$\mathcal{L} \left[ 8t^3 e^{4t} - 4\cos(5t) \right] = 8 \frac{4!}{(s - 4)^4} - 4 \frac{s}{s^2 + 25}.$$  

Other tricks: Shifting Theorem.

If you ever take the Laplace transform of a function $g(t)$ that is multiplied by an overall exponential $e^{at}$, like

$$f(t) = t^2 e^{4t}$$

then the first shifting theorem tells us that the Laplace transform is given by $G(s - a)$, the Laplace transform of the function $g(t)$ except that at the last minute we write $s - a$ every time we wanted to write $s$. In the above example,
\( a = 4 \) and \( g(t) = t^2 \). In this case, the answer is \( F(s) = \mathcal{L} \left[ t^2 e^{4t} \right] = \mathcal{L} \left[ t^2 \right] \), where in the end we write \( s - 4 \) instead of \( s \). Since
\[
\mathcal{L} \left[ t^2 \right] = \frac{2}{s^3},
\]
giving
\[
\mathcal{L} \left[ t^2 e^{4t} \right] = \frac{2}{(s - 4)^3}.
\]

Inverse transforms are a little trickier some times, but basically the same. You just find the correct entry in the table, or perhaps massage your function a bit to make it look like one. For example,
\[
F(s) = \frac{1}{\sqrt{s - 3}}
\]
This is not in the table – although it is close to
\[
F(s) = \sqrt{\frac{\pi}{s}}
\]
whose solution is \( f(t) = 1/\sqrt{t} \). To be useful, we need the denominator to be \( s - 3 \), which we can do with the first shifting theorem; and we need the numerator to be \( \sqrt{\pi} \), which can do by just multiplying. So this gives
\[
\frac{1}{\sqrt{\pi}} \mathcal{L}^{-1} \left[ \frac{\sqrt{\pi}}{\sqrt{s - 3}} \right] = \frac{1}{\sqrt{\pi}} \sqrt{\pi} e^{-3t}
\]

**Second Shifting Theorem**

Now look at the Heaviside step function \( H(t - a) \), which goes from 0 to 1 discontinuously at \( t = a \). Think of it like turning on a switch. The second shifting theorem deals with shifted time:
\[
\mathcal{L} \left[ H(t - a) f(t - a) \right] = e^{-as} F(s),
\]
which can be read both ways. First off – if you have an exponential \( e^{-as} \) in the Laplace transformed function, then you know you will get an answer that contains \( a \) the inverse Laplace-transformed \( f(t) \), but with by the time shifted by \( t \rightarrow (t - a) \), and also that the final answer will be multiplied by \( H(t - a) \). Second – If you have a heaviside function and want to take the Laplace transform, then write the rest of the function \( f(t) \) in terms of a shifted variable \( t - a \). You can do this by replacing \( t \) with \( t - a \), and then adding it back again. For example,
\[
\mathcal{L} \left[ H(t - a)(t + 3) \right] = \mathcal{L} \left[ H(t - a)(t - a + 3 + a) \right] = \mathcal{L} \left[ H(t - a)f(t - a) \right],
\]
where
\[
f(t) = t + 3 + a
\]
Now, we use the second shifting theorem – simply calculate the L. T. of \( f(t) \), then multiply by \( e^{-as} \):
\[
F(s) = e^{-as} \left( \frac{1}{s^2} + \frac{3 + a}{s} \right).
\]

Other important things: A pulse is given by
\[
y(t) = AH(t - a) - BH(t - b).
\]
and a delta function *impulse* occurs when an infinitely fast pulse (but finite area underneath) is used to 'kick' the system.
\[
\delta(t - a) = \lim_{\epsilon \to 0} \frac{H(t - a) - H(t - a - \epsilon)}{\epsilon}
\]
Lastly – why would you want to do this? Laplace transform derivatives – can solve linear ODE’s this way. We will see the initial values enter naturally. Consider
\[ \dot{y} + 8y = 8 \quad (73) \]
subject to \( y(0) = 8 \). You can solve this as we did above by using an integrating factor, or plugging in \( y = e^{\lambda t} \) for the homogeneous solution, then treat the inhomogeneous bit with undetermined coefficients or something. Here we are going to use Laplace Transforms. Laplace transform the whole equation:

\[
\mathcal{L}[\dot{y}] = sY(s) - y(0) \\
\mathcal{L}[8y] = 8Y(s) \\
\mathcal{L}[8] = 8/s
\]

Put them together and you get
\[ sY(s) - y(0) + 8Y(s) = \frac{8}{s} \quad (77) \]
which can be solved to give
\[ (s + 8)Y(s) = \frac{8}{s} + y(0) \quad (78) \]
or
\[ Y(s) = \frac{8}{s(s + 8)} + \frac{y(0)}{s + 8} \quad (79) \]
These can then be inverted individually. Note first that \( y(0) = 8 \), so
\[ \mathcal{L}^{-1} \left[ \frac{y(0)}{s + 8} \right] = 8e^{-8t} \quad (80) \]
and
\[ 8\mathcal{L}^{-1} \left[ \frac{1}{s(s + 8)} \right] = \frac{8}{s + 8} \left( e^{0} - e^{-8t} \right) \quad (81) \]
Adding these together gives
\[ y(t) = 1 + 7e^{-8t}. \quad (82) \]

One last example: consider a set of reactions of chemicals \( a \) and \( b \).
\[ \dot{b} = -b + 2a + 3 \quad (83) \]
\[ \dot{a} = b - 2a \quad (84) \]
with \( a(0) = b(0) = 0 \). Here note 3 is an inhomogeneous term – we stir in some amount of \( b \) every minute... Take the laplace transform of both equations:

\[
sB = -B + 2A + 3/s \\
\mathcal{L}[\dot{A}] = B - 2A
\]

Now, solve the second equation for \( A \):
\[ A = \frac{B}{s + 2} \quad (87) \]
and plug into the first eqn:
\[ \left( s + 1 - \frac{2}{s + 2} \right) B = \frac{3}{s} \quad (88) \]
Some algebra gives
\[ B(s) = \frac{3(s+2)}{s^2(s+3)} = \frac{2}{s^2} + \frac{1}{3s} - \frac{1}{3(3+s)} \]  
(89)
\[ A(s) = \frac{3}{s^2(s+3)} = \frac{1}{s^2} + \frac{1}{3s} + \frac{1}{3(3+s)}. \]  
(90)
(Note I used the Mathematica function ‘Apart’ to get the partial fraction decomposition.) This can be inverted to give
\[ b(t) = 2t + \frac{1}{3} (1 - e^{-3t}) \]  
(91)
\[ a(t) = t + \frac{1}{3} (e^{-3t} - 1). \]  
(92)

**Convolutions.** In class, we derived the convolution theorem. The details of how to do so are not as important as knowing how to use it. A convolution occurs when you have the product of two functions, say \( F(s) \) and \( G(s) \). The convolution formula says
\[ \mathcal{L}^{-1} [F(s)G(s)] = \int_0^t f(t-\tau)g(\tau)d\tau. \]  
(93)
where \( f(t) \) and \( g(t) \) are \( \mathcal{L}^{-1} [F(s)] \) and \( \mathcal{L}^{-1} [G(s)] \), respectively. In other words, if you have two Laplace Transforms multiplying each other, than to find the inverse transform, you should first find the inverse transforms of both (i.e. \( f(t) \) and \( g(t) \)), then replace \( t \) by \( t-\tau \) in \( f(t) \), and replace \( t \) by \( \tau \) in \( g(t) \), and then compute the integral.

For example, say we want to find the inverse Laplace Transform \( \mathcal{L}^{-1} [1/s^4] \) in a particularly silly way: by considering \( F(s) = 1/s^2 \) and \( G(s) = 1/s^2 \), whose respective inverse transforms are \( f(t) = t \) and \( g(t) = t \). In this case,
\[ \mathcal{L}^{-1} [1/s^4] = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t (t-\tau)\tau d\tau = \frac{t\tau^2}{2} - \frac{\tau^3}{3} \bigg|_0^t = \frac{t^3}{6} \]  
(94)
as we could have done directly from the very beginning.

Convolutions are useful in solving inhomogeneous ODEs – what if you solve an ODE where the inhomogeneity is left ‘general’? For example, in class we did
\[ \frac{dc}{dt} + c = g(t) \]  
(95)
whose Laplace Transform is
\[ sC(s) - c(0) + C(s) = G(s) \]  
(96)
giving
\[ C(s) = \frac{c(0)}{1+s} + \frac{G(s)}{1+s}. \]  
(97)
What we really want, in the end, is \( c(t) \):
\[ c(t) = c(0)e^{-t} + \mathcal{L}^{-1} \left[ \frac{G(s)}{1+s} \right]. \]  
(98)
If we had specified \( g(t) \), then we could have computed \( G(s) \) directly using Tables or Mathematica or whatever. We could then simply invert the above equation to give an explicit solution for \( c(t) \). However, we have left \( g(t) \) unspecified – it can literally be whatever function you want it to be. The convolution theorem allows you to actually compute the inverse, even for an arbitrary function – here \( F(s) = 1/(1+s) \), so that \( f(t) = e^{-t} \), and \( G(s) \) and \( g(t) \) are unknown. Using the convolution theorem,
\[ \mathcal{L}^{-1} \left[ \frac{G(s)}{1+s} \right] = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t e^{-(t-\tau)}g(\tau)d\tau, \]  
(99)
so that
\[ c(t) = c(0)e^{-t} + \int_0^t e^{-(t-\tau)}g(\tau)d\tau. \]  
(100)
Now, you can put in any \( g(t) \) you like and compute \( c(t) \).