14
Design of Experiments with Several Factors

Reading Assignment: pp. 539-544, 559-564
• Suppose that an important variable $Y$ may depend on several independent variables, $X_1, X_2, \ldots, X_n$.

• **Key Issues:**
  - How do we determine which $\{X_i\}$ are most important?
  - How do we characterize their relative effects?

• **Important applications:**
  - Process development
  - Process improvement
14-1 Introduction

• An **experiment** is a test or series of tests.

• The **design** of an experiment plays a major role in the eventual solution of the problem.

• **Definitions for the experiments:**
  - **Response Variable:** The dependent variable $Y$
  - **Factors:** The independent variables being considered, the $\{X_i\}$.
  - **Levels:** Values of the factors being considered

• In a **factorial experimental design**, experimental trials (or runs) are performed at all combinations of the factor levels.
Question:
Why not simply vary one factor at a time and evaluate its effect on the response variable?

Answer:
This intuitive strategy may give very poor results.

Example: Batch reactor yield:
  Response variable: Yield (%)
  Factors: Reaction time and reaction temperature
Example: Batch Reactor Yield

Two factors: Reaction time and temperature

Note: Maximum yield occurs at $t = 1.7$ hr.

Figure 14-7 Yield versus reaction time with temperature constant at 155º F.
Example: Batch Reactor Yield

Figure 14-8 Yield versus temperature with reaction time constant at 1.7 hours.
Contour Plot for Batch Reactor Example

Contours represent constant values of yield in %.

Figure 14-9
Optimization experiment using the one-factor-at-a-time method.
14-2 Factorial Experiments

Definition

By a factorial experiment we mean that in each complete trial or replicate of the experiment all possible combinations of the levels of the factors are investigated.

<table>
<thead>
<tr>
<th>Table 14-1</th>
<th>A Factorial Experiment with Two Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factor A</strong></td>
<td><strong>Factor B</strong></td>
</tr>
<tr>
<td></td>
<td>$B_{low}$</td>
</tr>
<tr>
<td>$A_{low}$</td>
<td>10</td>
</tr>
<tr>
<td>$A_{high}$</td>
<td>30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 14-2</th>
<th>A Factorial Experiment with Interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factor A</strong></td>
<td><strong>Factor B</strong></td>
</tr>
<tr>
<td></td>
<td>$B_{low}$</td>
</tr>
<tr>
<td>$A_{low}$</td>
<td>10</td>
</tr>
<tr>
<td>$A_{high}$</td>
<td>30</td>
</tr>
</tbody>
</table>
Figure 14-3 Factorial Experiment, no interaction.
14-2 Factorial Experiments

Figure 14-4 Factorial Experiment, with interaction.
14-5. $2^k$ Factorial Designs: 
$k$ factors and 2 levels for each factor.

Initially, we will consider $k = 2$ (a $2^2$ design) with:

**Factors**: A and B.  **Levels**: (+) and (-)

*Figure 14-15* The $2^2$ factorial design.
2² Factorial Designs

The **main effect** of a factor is estimated by evaluating the corresponding values of response variable, \( y \). Let:

\[
A = \bar{y}_{A+} - \bar{y}_{A-} = \frac{a + ab}{2n} - \frac{b + (1)}{2n} = \frac{1}{2n} [a + ab - b - (1)]
\]

(14-11)
The main effect of a factor B is estimated by

\[ B = \bar{y}_{B+} - \bar{y}_{B-} \]

\[ = \frac{b + ab}{2n} - \frac{a + (1)}{2n} \]

\[ = \frac{1}{2n} [b + ab - a - (1)] \]

(14-12)
The AB interaction effect is estimated by

\[
AB = \frac{ab + (1)}{2n} - \frac{a + b}{2n} - \frac{1}{2n} [ab + (1) - a - b]
\]

(14-13)
14-5 2<sup>k</sup> Factorial Designs

14-5.1 2<sup>2</sup> Design

The quantities in brackets in Equations 14-11, 14-12, and 14-13 are called **contrasts**. For example, the $A$ contrast is

\[
\text{Contrast}_A = a + ab - b - (1)
\]

<table>
<thead>
<tr>
<th>Treatment Combination</th>
<th>Factorial Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$I$</td>
</tr>
<tr>
<td>(1)</td>
<td>+</td>
</tr>
<tr>
<td>$a$</td>
<td>+</td>
</tr>
<tr>
<td>$b$</td>
<td>+</td>
</tr>
<tr>
<td>$ab$</td>
<td>+</td>
</tr>
</tbody>
</table>
**14-5 2^k Factorial Designs**

**14-5.1 2^2 Design**

Contrasts are used in calculating both the effect estimates and the sums of squares for $A$, $B$, and the $AB$ interaction. The sums of squares formulas are

\[
SS_A = \frac{[a + ab - b - (1)]^2}{4n}
\]

\[
SS_B = \frac{[b + ab - a - (1)]^2}{4n}
\]

\[
SS_{AB} = \frac{[ab + (1) - a - b]^2}{4n}
\]

(14-14)
## ANOVA Table for Example 14-3

### Table 14-14  Analysis of Variance for the Epitaxial Process Experiment

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$f_0$</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (deposition time)</td>
<td>2.7956</td>
<td>1</td>
<td>2.7956</td>
<td>134.40</td>
<td>7.07 E-8</td>
</tr>
<tr>
<td>$B$ (arsenic flow)</td>
<td>0.0181</td>
<td>1</td>
<td>0.0181</td>
<td>0.87</td>
<td>0.38</td>
</tr>
<tr>
<td>$AB$</td>
<td>0.0040</td>
<td>1</td>
<td>0.0040</td>
<td>0.19</td>
<td>0.67</td>
</tr>
<tr>
<td>Error</td>
<td>0.2495</td>
<td>12</td>
<td>0.0208</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>3.0672</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Note:

The $f_0$ value for effect $i$ is calculated as:

$$f_{0i} = \frac{MS_i}{MS_E}$$

$SS_E$ is calculated by subtracting all of the individual $SS_i$ from $SS_T$. 
14-5 $2^k$ Factorial Designs

14-5.4 Additional Center Points to a $2^k$ Design

A potential concern in the use of two-level factorial designs is the assumption of the linearity in the factor effect. Adding center points to the $2^k$ design will provide protection against curvature as well as allow an independent estimate of error to be obtained.
14-5 2^k Factorial Designs

14-5.2 2^k Design for k ≥ 3 Factors

Figure 14-20 The 2^3 design.
Figure 14-21 Geometric presentation of contrasts corresponding to the main effects and interaction in the $2^3$ design. (a) Main effects. (b) Two-factor interactions. (c) Three-factor interaction.