Frequency Response Analysis

Sinusoidal Forcing of a First-Order Process

For a first-order transfer function with gain $K$ and time constant $\tau$, the response to a general sinusoidal input, $x(t) = A \sin \omega t$ is:

$$y(t) = \frac{KA}{\omega^2 \tau^2 + 1} \left( \omega \tau e^{-t/\tau} - \omega \tau \cos \omega t + \sin \omega t \right) \quad (5-25)$$

Note that $y(t)$ and $x(t)$ are in deviation form. The long-time response, $y_\ell(t)$, can be written as:

$$y_\ell(t) = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \sin (\omega t + \varphi) \text{ for } t \to \infty \quad (13-1)$$

where:

$$\varphi = -\tan^{-1}(\omega \tau)$$
Figure 13.1 Attenuation and time shift between input and output sine waves ($K=1$). The phase angle $\varphi$ of the output signal is given by $\varphi = -\frac{\text{Time shift}}{P \times 360^\circ}$, where $\Delta t$ is the (period) shift and $P$ is the period of oscillation.
Chapter 13

Frequency Response Characteristics of a First-Order Process

For \( x(t) = A \sin \omega t \), \( y_\ell(t) = \hat{A} \sin(\omega t + \phi) \) as \( t \to \infty \) where:

\[
\hat{A} = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \quad \text{and} \quad \phi = -\tan^{-1}(\omega \tau)
\]  

1. The output signal is a sinusoid that has the same frequency, \( \omega \), as the input signal, \( x(t) = A \sin \omega t \).

2. The amplitude of the output signal, \( \hat{A} \), is a function of the frequency \( \omega \) and the input amplitude, \( A \):

\[
\hat{A} = \frac{KA}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-2)
\]

3. The output has a phase shift, \( \phi \), relative to the input. The amount of phase shift depends on \( \omega \).
Dividing both sides of (13-2) by the input signal amplitude $A$ yields the amplitude ratio (AR)

$$AR = \frac{\hat{A}}{A} = \frac{K}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-3a)$$

which can, in turn, be divided by the process gain to yield the normalized amplitude ratio (AR$_N$)

$$AR_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \quad (13-3b)$$
Shortcut Method for Finding the Frequency Response

The shortcut method consists of the following steps:

**Step 1.** Set \( s = j\omega \) in \( G(s) \) to obtain \( G(j\omega) \).

**Step 2.** Rationalize \( G(j\omega) \); We want to express it in the form:

\[
G(j\omega) = R + jI
\]
where \( R \) and \( I \) are functions of \( \omega \). Simplify \( G(j\omega) \) by multiplying the numerator and denominator by the complex conjugate of the denominator.

**Step 3.** The amplitude ratio and phase angle of \( G(s) \) are given by:

\[
\text{AR} = \sqrt{R^2 + I^2}
\]

\[
\varphi = \tan^{-1}(I/R)
\]

Memorize ⇒
**Example 13.1**

Find the frequency response of a first-order system, with

\[ G(s) = \frac{1}{\tau s + 1} \quad \text{(13-16)} \]

**Solution**

First, substitute \( s = j\omega \) in the transfer function

\[ G(j\omega) = \frac{1}{j\omega \tau + 1} = \frac{1}{j\omega \tau + 1} \quad \text{(13-17)} \]

Then multiply both numerator and denominator by the complex conjugate of the denominator, that is, \(-j\omega \tau + 1\)

\[ G(j\omega) = \frac{-j\omega \tau + 1}{(j\omega \tau + 1)(-j\omega \tau + 1)} = \frac{-j\omega \tau + 1}{\omega^2 \tau^2 + 1} \]

\[ = \frac{1}{\omega^2 \tau^2 + 1} + j \frac{-\omega \tau}{\omega^2 \tau^2 + 1} = R + jI \quad \text{(13-18)} \]
where:

\[ R = \frac{1}{\omega^2 \tau^2 + 1} \]  \hspace{1cm} (13-19a)

\[ I = \frac{-\omega \tau}{\omega^2 \tau^2 + 1} \]  \hspace{1cm} (13-19b)

From Step 3 of the Shortcut Method,

\[ AR = \sqrt{R^2 + I^2} = \sqrt{\left(\frac{1}{\omega^2 \tau^2 + 1}\right)^2 + \left(\frac{-\omega \tau}{\omega^2 \tau^2 + 1}\right)^2} \]

or

\[ AR = \frac{\sqrt{\left(1 + \omega^2 \tau^2\right)^2}}{\sqrt{(\omega^2 \tau^2 + 1)^2}} = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \]  \hspace{1cm} (13-20a)

Also,

\[ \varphi = \tan^{-1}\left(\frac{I}{R}\right) = \tan^{-1}(-\omega \tau) = -\tan^{-1}(\omega \tau) \]  \hspace{1cm} (13-20b)
Consider a complex transfer function $G(s)$,

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\ldots}{G_1(s)G_2(s)G_3(s)\ldots}$$  \hspace{1cm} (13-22)

Substitute $s=j\omega$,

$$G(j\omega) = \frac{G_a(j\omega)G_b(j\omega)G_c(j\omega)\ldots}{G_1(j\omega)G_2(j\omega)G_3(j\omega)\ldots}$$  \hspace{1cm} (13-23)

From complex variable theory, we can express the magnitude and angle of $G(j\omega)$ as follows:

$$|G(j\omega)| = \left|\frac{G_a(j\omega)}{G_1(j\omega)}\right|\left|\frac{G_b(j\omega)}{G_2(j\omega)}\right|\left|\frac{G_c(j\omega)}{G_3(j\omega)}\right|\ldots$$  \hspace{1cm} (13-24a)

$$\angle G(j\omega) = \angle G_a(j\omega) + \angle G_b(j\omega) + \angle G_c(j\omega) + \ldots - [\angle G_1(j\omega) + \angle G_2(j\omega) + \angle G_3(j\omega) + \ldots]$$  \hspace{1cm} (13-24b)
Bode Diagrams

- A special graph, called the *Bode diagram* or *Bode plot*, provides a convenient display of the frequency response characteristics of a transfer function model. It consists of plots of AR and $\varphi$ as a function of $\omega$.

- Ordinarily, $\omega$ is expressed in units of radians/time.

**Bode Plot of A First-order System**

Recall:

$$AR_N = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega \tau)$$

- **At low frequencies** ($\omega \to 0$ and $\omega \tau \ll 1$):
  $$AR_N = 1 \quad \text{and} \quad \varphi = 0$$

- **At high frequencies** ($\omega \to \infty$ and $\omega \tau \gg 1$):
  $$AR_N = \frac{1}{\omega \tau} \quad \text{and} \quad \varphi = -90^\circ$$
Figure 13.2 Bode diagram for a first-order process.
• Note that the asymptotes intersect at $\omega = \omega_b = 1/\tau$, known as the break frequency or corner frequency. Here the value of $AR_N$ from (13-21) is:

$$AR_N(\omega = \omega_b) = \frac{1}{\sqrt{1+1}} = 0.707 \quad (13-30)$$

• Some books and software defined AR differently, in terms of decibels. The amplitude ratio in decibels $AR_d$ is defined as

$$AR_d = 20 \log AR \quad (13-33)$$
**Integrating Elements**

The transfer function for an integrating element was given in Chapter 5:

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s} \quad \text{(5-34)}
\]

\[
\text{AR} = |G(j\omega)| = \left| \frac{K}{j\omega} \right| = \frac{K}{\omega} \quad \text{(13-34)}
\]

\[
\varphi = \angle G(j\omega) = \angle K - \angle(\infty) = -90^\circ \quad \text{(13-35)}
\]

**Second-Order Process**

A general transfer function that describes any underdamped, critically damped, or overdamped second-order system is

\[
G(s) = \frac{K}{\tau^2 s^2 + 2\zeta \tau s + 1} \quad \text{(13-40)}
\]
Substituting \( s = j\omega \) and rearranging yields:

\[
AR = \frac{K}{\sqrt{(1 - \omega^2 \tau^2)^2 + (2\omega \tau \zeta)^2}}
\]  \hspace{1cm} (13-41a)

\[
\phi = \tan^{-1}\left[ \frac{-2\omega \tau \zeta}{1 - \omega^2 \tau^2} \right]
\]  \hspace{1cm} (13-41b)

Figure 13.3 Bode diagrams for second-order processes.
Time Delay

Its frequency response characteristics can be obtained by substituting \( s = j\omega \),

\[
G(j\omega) = e^{-j\omega \theta} \quad (13-53)
\]

which can be written in rational form by substitution of the Euler identity,

\[
G(j\omega) = e^{-j\omega \theta} = \cos \omega \theta - j \sin \omega \theta \quad (13-54)
\]

From (13-54)

\[
\text{AR} = |G(j\omega)| = \sqrt{\cos^2 \omega \theta + \sin^2 \omega \theta} = 1 \quad (13-55)
\]

\[
\phi = \angle G(j\omega) = \tan^{-1} \left( -\frac{\sin \omega \theta}{\cos \omega \theta} \right)
\]

or

\[
\phi = -\omega \theta \quad (13-56)
\]
Figure 13.6 Bode diagram for a time delay, $e^{-\theta s}$. 
Figure 13.7 Phase angle plots for $e^{-\theta s}$ and for the 1/1 and 2/2 Padé approximations ($G_1$ is 1/1; $G_2$ is 2/2).
Consider a process zero term,

\[ G(s) = K(s\tau + 1) \]

Substituting \( s=j\omega \) gives

\[ G(j\omega) = K(j\omega\tau + 1) \]

Thus:

\[ \text{AR} = \left| G(j\omega) \right| = K\sqrt{\omega^2\tau^2 + 1} \]

\[ \phi = \angle G(j\omega) = + \tan^{-1}(\omega\tau) \]

**Note:** In general, a multiplicative constant (e.g., \( K \)) changes the AR by a factor of \( K \) without affecting \( \phi \).
**Frequency Response Characteristics of Feedback Controllers**

**Proportional Controller.** Consider a proportional controller with positive gain

\[ G_c(s) = K_c \]  \hspace{1cm} (13-57)

In this case \( |G_c(j\omega)| = K_c \), which is independent of \( \omega \). Therefore,

\[ AR_c = K_c \]  \hspace{1cm} (13-58)

and

\[ \phi_c = 0^\circ \]  \hspace{1cm} (13-59)
**Proportional-Integral Controller.** A proportional-integral (PI) controller has the transfer function (cf. Eq. 8-9),

\[ G_c(s) = K_c \left( 1 + \frac{1}{\tau_I s} \right) = K_c \left( \frac{\tau_I s + 1}{\tau_I s} \right) \quad (13-60) \]

Substitute \( s = j\omega \):

\[ G_c(j\omega) = K_c \left( 1 + \frac{1}{\tau_I j\omega} \right) = K_c \left( \frac{j\omega\tau_I + 1}{j\omega\tau_I} \right) = K_c \left( 1 - \frac{1}{\tau_I \omega} j \right) \]

Thus, the amplitude ratio and phase angle are:

\[ \text{AR}_c = |G_c(j\omega)| = K_c \sqrt{1 + \frac{1}{(\omega\tau_I)^2}} = K_c \frac{\sqrt{(\omega\tau_I)^2 + 1}}{\omega\tau_I} \quad (13-62) \]

\[ \phi_c = \angle G_c(j\omega) = \tan^{-1}\left(-1/\omega\tau_I\right) = \tan^{-1}\left(\omega\tau_I\right) - 90^\circ \quad (13-63) \]
Figure 13.9 Bode plot of a PI controller, $G_c(s) = 2\left(\frac{10s+1}{10s}\right)$
**Ideal Proportional-Derivative Controller.** For the ideal proportional-derivative (PD) controller (cf. Eq. 8-11)

\[ G_c(s) = K_c (1 + \tau_D s) \quad (13-64) \]

The frequency response characteristics are similar to those of a LHP zero:

\[ AR_c = K_c \sqrt{(\omega \tau_D)^2 + 1} \quad (13-65) \]

\[ \varphi = \tan^{-1}(\omega \tau_D) \quad (13-66) \]

**Proportional-Derivative Controller with Filter.** The PD controller is most often realized by the transfer function

\[ G_c(s) = K_c \left( \frac{\tau_D s + 1}{\alpha \tau_D s + 1} \right) \quad (13-67) \]
Figure 13.10 Bode plots of an ideal PD controller and a PD controller with derivative filter.

Idea: \( G_c(s) = 2(4s + 1) \)

With Derivative Filter:
\[
G_c(s) = 2\left(\frac{4s + 1}{0.4s + 1}\right)
\]


**PID Controller Forms**

*Parallel PID Controller.* The simplest form in Ch. 8 is

\[
G_c(s) = K_c \left( 1 + \frac{1}{\tau_1 s} + \tau_D s \right)
\]

*Series PID Controller.* The simplest version of the series PID controller is

\[
G_c(s) = K_c \left( \frac{\tau_1 s + 1}{\tau_1 s} \right) (\tau_D s + 1) \quad (13-73)
\]

*Series PID Controller with a Derivative Filter.*

\[
G_c(s) = K_c \left( \frac{\tau_1 s + 1}{\tau_1 s} \right) \left( \frac{\tau_D s + 1}{\alpha \tau_D s + 1} \right)
\]
Figure 13.11 Bode plots of ideal parallel PID controller and series PID controller with derivative filter ($\alpha = 1$).

Idea parallel:

$$G_c(s) = 2 \left( 1 + \frac{1}{10s} + 4s \right)$$

Series with Derivative Filter:

$$G_c(s) = 2 \left( \frac{10s + 1}{10s} \right) \left( \frac{4s + 1}{0.4s + 1} \right)$$
Consider the transfer function

\[
G(s) = \frac{1}{2s + 1}
\]

(13-76)

with

\[
AR = |G(j\omega)| = \frac{1}{\sqrt{(2\omega)^2 + 1}}
\]

(13-77a)

and

\[
\varphi = \angle G(j\omega) = -\tan^{-1}(2\omega)
\]

(13-77b)
Figure 13.12 The Nyquist diagram for $G(s) = \frac{1}{2s + 1}$ plotting $\text{Re}(G(j\omega))$ and $\text{Im}(G(j\omega))$. 
Figure 13.13 The Nyquist diagram for the transfer function in Example 13.5:

\[ G(s) = \frac{5(8s + 1)e^{-6s}}{(20s + 1)(4s + 1)} \]